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ratio a/b has the *single value* $4/3$ (when in lowest terms); the rule holds for this but for no other value.

Corollary: Generalizing by substituting the rational fraction t/u in place of $1/2$ in the boys' rule, we get:

$$h = \frac{t}{u}a + b = \frac{ta + ub}{u};$$

then

$$h^2 = a^2 + b^2 = \frac{t^2a^2 + 2tuab + u^2b^2}{u^2}.$$

Whence $u^2a^2 - t^2a^2 = 2tuab$, and $a(u^2 - t^2) = 2tub$.

Consequently

$$\frac{b}{a} = \frac{u^2 - t^2}{2tu}.$$

The two legs are therefore in the proportion: $(u^2 - t^2) : 2tu$. Let us take exactly the values $u^2 - t^2$ and $2tu$ for the sides.

Hence the hypotenuse $= \sqrt{(u^2 - t^2)^2 + (2tu)^2} = u^2 + t^2$. Therefore

$$a = u^2 - t^2; \qquad b = 2tu; \qquad h = u^2 + t^2, \tag{1}$$

where t and u are integers, to be chosen at pleasure, (except that $u > t$, since a is to be positive). More generally we may write

$$a = c(u^2 + t^2); \qquad b = c(2tu); \qquad h = c(u^2 + t^2), \tag{2}$$

which is the well-known formula for Pythagorean triangles in its most general form. The particular case, $c = 1$, gives the primitive types.

RECENT PUBLICATIONS.

REVIEWS.

HISTORY OF THE THEORY OF NUMBERS.

History of the Theory of Numbers. Volume I: *Divisibility and Primality.* By LEONARD EUGENE DICKSON. Carnegie Institution of Washington, 1919, publication number 256, Vol. I. 14 + 486 pages. Price, in paper, \$7.50; in cloth, \$8.00.

There are at least three great ends to serve by the history of science: to enrich the general culture and intellectual life of cultivated people; to enable a scientific worker quickly to orient himself in a chapter of science so as to proceed most readily to a detailed mastery of its literature; to enable a scientific worker to ascertain with completeness what has already been attained in a given subject. To serve any one of these three purposes requires a very different treatment from that needed in the case of the others. It is probably impossible to contribute satisfactorily at the same time and by the same work to any two of them. The present literature of each class is altogether inadequate in the case of science in general and in each particular discipline.

It is hard to say which of these three aspects of the need of scientific history presses hardest upon our attention. Each one would probably decide in favor of that which most vitally affects his own particular interest. Probably the last of the three purposes named is realized in the fewest instances. This work of Dickson, of which the first volume is now before us, embodies an attempt to serve this purpose adequately so far as the theory of numbers is concerned. The work is arranged so as to be suitable not only to the needs of the researcher but also to the interests of the amateur who studies number theory in the spirit of play and with restful delight.

From the preface we quote: "The theory of numbers is especially entitled to a separate history on account of the great interest which has been taken in it continuously through the centuries from the time of Pythagoras, an interest shared on the one extreme by nearly every noted mathematician and on the other extreme by numerous amateurs attracted by no other part of mathematics. This history aims to give an adequate account of the entire literature of the theory of numbers. The first volume presents in twenty chapters the material relating to divisibility and primality. The concepts, results, and authors cited are so numerous that it seems appropriate to present here an introduction which gives for certain chapters an account in untechnical language of the main results in their historical setting, and for the remaining chapters the few remarks sufficient to clearly characterize the nature of their contents."

To give an adequate account of the entire literature of so vast a subject and one of such long history as the theory of numbers is an undertaking of enormous magnitude; and it is carried through in this work with a marvelous success in the presence of which one must pause in admiration. Henceforth this history will be indispensable to all investigators in the theory of numbers. No one will be justified in publishing his researches on this subject till he has compared them with the relevant chapters of this work. Frequent duplications, all too common in the past development of the subject, will have far less excuse for their continuance. A great deal of scientific energy will be saved through this storehouse of exact information about the development of the theory of numbers and the actual present state of knowledge.

As indicated already, the subject is well suited to the interests of amateurs in mathematics. The reviewer's private correspondence shows that not a few such people, interested in number theory, are to be found throughout our country (and presumably in other countries as well). Many of these are far separated from adequate scientific libraries. To such this book will now be indispensable. With it in hand, even though they are separated from library sources, they will be able to get a fair estimate of the novelty of their work. The book will certainly add greatly to the delight and the value of the work of such people. Since the theory of numbers is precisely that part of mathematics in which the amateur has the greatest opportunity of making a contribution of interest it may be hoped that one of the pleasing fruits of publishing such a work as the present will be the inspiration afforded to amateurs to pursue their work and to make known their discoveries in this field of fascinating interest which has inspired the enthusiasm of people of every century for some millenniums.

Where the detailed results are so numerous and are distributed widely through so varied a literature it is impossible that the first adequate research into the matter should uncover all the information and that no errors should be made in the presentation of what was found. Realizing this, the author has put in his preface the statement: "Readers are requested to . . . increase the usefulness of this work by sending corrections, notices of omissions, and abstracts of papers marked not available for report, for insertion in the concluding volume." It is to be hoped that this request may have wide notice and that every addition or correction found to be needed will be sent to the author at the University of Chicago. Besides several minor corrections which the reviewer is sending to the author in this way mention should now be made of the following changes needful to avoid confusion or annoyance on the reader's part: p. 131, l. 4, change "simple" to "prime power cyclic"; p. 381, l. 11, change k to m in two places; p. 403, l. 22, change $U_{2\ +1}$ to U_{2n+1}; p. 413, l. 13, read J. Perott[8] noted that, if $p_1, \cdots, p_n$ are all the primes and their product $\leqq N \cdots$.

Probably no one sufficiently interested in the theory of numbers to read this review will be satisfied without examining the volume itself. He will naturally begin with the ten-page preface or prefatory introduction, where we have a rapid review of some of the main features of the development of the theory covered by the present volume. Not a few people will wish that this had been more extended, especially since the immense amount of detail necessary in the body of the book makes it hard sometimes to select the salient matters. Necessarily the classification is not into rigidly separated portions so that matters more or less closely related will sometimes be found in different chapters. The subject index will be found to bring these together in a helpful way.

The book opens with a forty-eight page history of perfect, multiply perfect, and amicable numbers, a group of closely related topics which have engaged the attention of arithmeticians of every century of the Christian era. If the sum $\sigma(n)$ of the divisors of n is a multiple mn of n then n is said to be a multiply perfect number of multiplicity m. If $m = 2$, so that n is the sum of its aliquot divisors, we say that n is perfect. Twelve perfect numbers are known as such. The list of multiply perfect numbers is much larger. In comparing the two new multiply perfect numbers due to Cunningham and given on page 37 of Dickson's *History* with the table of Carmichael and Mason I discovered in the latter the error of taking 137561 [$= 151.911$] to be a prime. This requires the exclusion from their table of the 19 numbers which have the factor 19^4 (but not 19^5). But the two numbers of Cunningham show that if one of the numbers $C \cdot 19^2 \cdot 127$ and $C \cdot 19^4 \cdot 151 \cdot 911$ is multiply perfect and C is prime to $19 \cdot 127 \cdot 151 \cdot 911$ then the other of these numbers is also multiply perfect of the same multiplicity. From this principle and the table of Carmichael and Mason it is easy to find (in a manner now obvious) 47 additional multiply perfect numbers. Moreover I have found also another number of multiplicity 7:

$$2^{32}\cdot 3^{11}\cdot 5^{4}\cdot 7^{8}\cdot 11^{2}\cdot 13^{2}\cdot 17^{2}\cdot 19^{3}\cdot 23\cdot 31\cdot 37^{2}\cdot 61\cdot 67\cdot 71\cdot 73\cdot 89\cdot 181\cdot 307\cdot 1063\cdot 2141 \cdot 599479.$$

In Carmichael and Mason's list the first number containing the factor 2^{46} should have the factor 67 inserted into it. With these additions and corrections there are now 282 known perfect and multiply perfect numbers.

Two numbers m and n are said to be amicable if their sum is equal to the sum of the divisors of each. Sixty-eight such numbers are known of which fifty-nine are due to Euler, three having been discovered before his time and six since. We propose to extend the notion of amicable numbers (in a way different from those indicated on page 50 of Dickson's *History*) and to say that two numbers m and n are multiply amicable of multiplicity k if k times their sum is equal to the sum of the divisors of each, so that we have $\sigma(m) = \sigma(n) = k(m + n)$. The usual amicable numbers are those for which $k = 1$. That multiply amicable numbers exist is shown by the following examples for $k = 2$:

$$2^5\cdot 3^2\cdot 7\cdot 13\cdot \begin{cases} 5\cdot 11 \\ 71 \end{cases} \qquad 2^2\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot \begin{cases} 11\cdot 19 \\ 239 \end{cases}.$$

The last is one of eleven pairs of multiplicity 2 which may be obtained from this one by replacing the factor $2^2\cdot 7$ by any one of the eleven known perfect numbers other than 6. Other similar sets may also be made up by multiplying certain amicable odd numbers of Euler's list by perfect numbers. It would be rather interesting to have a systematic determination of multiply amicable numbers after the method employed by Euler for $k = 1$ and so well described by Dickson on pages 42–46. [A similar generalization of Dickson's amicable triples (p. 50) would also be of some interest.]

After a short chapter on the number and sum of divisors and certain related problems of Fermat and Wallis, we have the extensive third chapter (pp. 59–103) on Fermat's and Wilson's theorems, their generalizations and converses, and symmetric functions modulo p of the positive integers less than p and prime to p; and this is followed by a closely related short chapter on the residue of $(u^{p-1} - 1)/p$ modulo p. In the preface it is pointed out that it was while investigating perfect numbers that Fermat was led to the theorem which bears his name and which forms the basis of a large part of the theory of numbers. Numerous properties and generalizations of Euler's φ-function are given in the long chapter V (pp. 113–158). Owing to the frequent occurrence of the φ-function in many parts of number theory the reader should employ the index to find accounts of its use in other ways than in those mentioned in chapter V. In this connection special attention should be given to the chapters on Lucas's functions and on recurring sequences of integers. The great amount of important detail included in such major chapters as III and V is such as to render an adequate summary impossible.

After an elementary chapter dealing with periodic fractions and factors of $10^n \pm 1$ we have in VII (pp. 181–222) and VIII (pp. 223–262) two other of the more fundamental chapters. The first of these deals with primitive roots and binomial congruences; the second with higher congruences. Together they form one of the most valuable sections of the book. Following them we have a chapter of sixteen pages on divisibility of factorials and multinomial coefficients. Then comes chapter X (pp. 279–325) on the sum and number of divisors of an integer and several closely related topics. Many of the results summarized here are closely connected with or are derived by means of various functional relations, sometimes associated with functions constructed for such use and at other times having to do with those which give rise incidentally to such applications. As in the case of other important chapters, its content is too vast for summarizing.

The next five chapters have to do with minor matters: miscellaneous theorems of divisibility, greatest common divisor, least common multiple; criteria for divisibility by a given number; factor tables, lists of primes; methods of factoring; Fermat numbers $F_n = 2^{2^n} + 1$. These chapters are necessary to completeness; more than that, some of the matter recorded still has its fascinating appeal.

In chapters XVI and XVII (pp. 381–411) we have an account of the factors of $a^n \pm b^n$, recurring series and Lucas's functions. The more or less fragmentary results here gathered together are of interest because of their intimate connection with the factorization of large numbers and the identification of large primes. Here the extent of our knowledge is so clearly marked by a definite line of cleavage between the known and the unknown, and the former is so clearly a mere fragment of the latter, that we contemplate our present state of knowledge with much the same feeling as that of the fabled Tantalus. The general theory of recurrent sequences of integers deserves a more penetrating analysis than it has yet received. It appears to be destined to bring out deep-lying connections among important parts of the theory of divisibility and primality. One who would go into this field for research could readily orient himself in it by means of Dickson's *History* and the two volumes (especially the second) of Bachmann's *Niedere Zahlentheorie.*

In Chapter XVIII (pp. 413–440), on the theory of prime numbers, we have in one respect, a departure from the plan followed in the others. That part of the subject covered by Landau's *Verteilung der Primzahlen* is passed over with a reference to its treatment in that work, particularly in the historical sections. Such other matter as pertains to the theory of primes is adequately treated in this chapter. To the various proofs that an infinitude of primes exists may be added the following: Since for a positive integer k we have $2^{2^n} \equiv -1$ or $(2^{2^n})^{2^k} \equiv (-1)^{2^k}$ or $2^{2^{n+k}} + 1 \equiv 2 \bmod 2^{2^n} + 1$ it follows (as is well known) that no two numbers of the form $2^{2^n} + 1$ have a common prime factor. Hence there are at least as many primes as numbers of this form and hence an infinite number.

One should consult the index for references to other tests for primes than those given in this chapter.

Chapter XIX (pp. 441–451) deals with a very interesting range of material under the heading: inversion of functions, Möbius's function $\mu(n)$, numerical integrals and derivatives. Important facts relating to inversion appear in other chapters and may be found through use of the index. Some of the inversion properties are very remarkable. A further systematic study of inversion would probably lead to novel results of interest. The relation $F(m) = \Sigma f(d)$, where d ranges over all the divisors of m, may be expressed by means of a Stieltjes integral in the form

$$F(m) = \int_0^m f(t) d_t \rho(m, t),$$

when $\rho(m, t)$ denotes the number of divisors of m not greater than t and $f(t)$ is a continuous function of t taking integral values when t is integral. The number-theoretic solution expressing f in terms of F leads to a solution in a special case of the foregoing Stieltjes integral equation. Other inversion problems may likewise be expressed by means of a numerical Stieltjes integral equation of the above or similar type. Is it likely that the inversion of numerical functions has already yielded all the results which would be brought to light by a systematic study of the inversion of numerical Stieltjes integrals formed in a manner now obvious, if these should be studied systematically under the guidance afforded by the existing extensive theory of ordinary integral equations?

The volume closes with thirteen pages on properties of the digits of numbers, at the same time one of the most elementary and one of the least interesting chapters of the book.

Not a few conjectured or empirical theorems are mentioned at various places throughout the work. These have interest as suggesting unanswered questions and as pointing out apparently simple matters which we are not at present able to treat satisfactorily. In this connection it is interesting to quote from Gauss: "The most beautiful theorems of the higher arithmetic have this characteristic that they are easily discovered by induction while their proof lies concealed and can be brought out only by a deeply penetrating investigation." And he adds that precisely this constitutes one of the greatest charms in number theory investigation.

Of the conjectured (unproved but not disproved) theorems in the present volume we call attention to the following:

1. If $2^n - 1$ is a prime p then $2^p - 1$ is a prime (pp. 22 and 24). This theorem, if true and proved, would yield an infinite sequence of identified primes; no such sequence is known at present.

2. If n is a prime of the form $24x + 11$ [$24x + 23$] and if $2^n - 1$ is composite, the least factor is of the form $24y + 23$ [$48y + 47$] (pp. 29, 30, 31).

3. If p is any number and a any divisor of $2^p - 1$, $a = 8m \pm 1$ not being of the form $2^n - 1$, then $2^a - 1$ is composite (p. 30).

4. Let $s(n) \equiv s^1(n)$ denote the sum of the divisors of n less than n and form $s^k(n) = s\{s^{k-1}(n)\}$. The set of numbers $n, s(n), s^2(n), \cdots$ may form a periodic cycle. When they do not it is conjectured that the set contains a prime and hence terminates (pp. 48–50).

5. Every prime $an^2 + p$ has a as a primitive root if a is a primitive root of p and p is a prime between $a/2$ and a (p. 186).

6. Let N be the greatest common divisor of all integers represented by a polynomial $f(x)$ with integral coefficients without a common factor. It is conjectured that $f(x)/N$ represents an infinitude of primes when $f(x)$ is irreducible (p. 333).

7. Every number of the following sequence is a prime (p. 376):

$$2 + 1, \qquad 2^2 + 1, \quad 2^{2^2} + 1, \qquad 2^{2^{2^2}} + 1, \quad \cdots.$$

8. Dickson asked if $a_in + b_i$ $(i = 1, \cdots, m)$ represent an infinitude of sets of primes when certain named necessary conditions are satisfied (p. 417).

9. Every even integer is a sum of two primes (pp. 421–424). On pp. 424 and 425 are several related theorems: every odd number is the sum of an odd square and the double of a prime of the form $4n + 1$; every prime $4n - 1$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$ [from which it would follow readily that every integer is a sum of four squares]; every even number is the difference of two (consecutive) primes in an infinitude of ways; every odd number greater than 3 is of each of the forms $p_1 + 2p_2$, $p_1 - 2p_2$, $2p_1 - p_2$, where p_1 and p_2 are primes; every multiple of 6 is the difference of two primes of the form $6n + 1$.

10. No three successive primes are in arithmetical progression unless one of them is 3 (p. 425).

11. Is there an unlimited number of sets of five consecutive odd integers of which four are prime (p. 426)?

12. If $n > 1$ there is at least one prime in each of the intervals $n(n - 1)$ to n^2, n^2 to $n(n + 1)$ (p. 435).

13. At least four primes lie between the squares of two consecutive primes each greater than 3 (p. 436).

14. The $(2m + 1)$th prime in order of magnitude (unity being counted as a prime) can be composed by addition and subtraction of all the smaller primes each taken once; the $(2m)$th prime can be composed similarly, except that the next earlier prime is doubled (p. 436).

It would be of interest to have some of these conjectured theorems verified for a considerable range of values; it is a service which may well be rendered by an amateur in number theory.

I have verified for primes less than 1300 the conjecture that a prime of the form $4n + 3$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$; the number of representations is sometimes large and seems to have a tendency to increase with increasing primes $4n + 3$. I have also verified up to 1300 that primes of the form $16n + 5$ [$36n + 7$] may be represented in the form $4p + q$ [$6p + q$] where p and q are primes of the form $4k + 1$.

A work of such extent and importance as the volume under review cannot be adequately described in a short article. To give a summary is impossible, since its 500 pages are themselves a summary in a very compact but lucid style of the results of many hundred memoirs. It is a piece of work for which one cannot find a parallel in the whole of scientific history. In completeness it goes far beyond anything before attempted for a wide range of material; and the work is done with a skill and accuracy which excite admiration. All persons interested in the theory of numbers are laid under a deep debt of gratitude to Professor Dickson.

R. D. CARMICHAEL.

NOTES.

Teubner announces the publication (1918) of a *Lehrbuch der Differential- und Integralrechnung und ihre Anwendungen* in two volumes, by R. Fricke.

The *Journal of the Michigan Schoolmasters' Club* (Ann Arbor, 1919) contains the proceedings of the fifty-third meeting, May 28–29, 1918. These include the following papers: "Group recitations in mathematics" by L. D. Wines, 82–86; "The value of measurements to high school mathematics" by S. A. Courtis, 87–94.

The following new English books are announced: H. Lamb, *Infinitesimal Calculus,* third edition (Cambridge University Press)—L. Silberstein, *Elements of Vector Algebra* (Longmans)—F. Cajori, *History of the Theories of Limits and Fluxions from Newton to Woodhouse* (Open Court)—H. E. J. Curzon, *Elementary Mathematics,* I and II.

W. Ahrens has published (Berlin, 1918) a new book on mathematical recreations, entitled *Altes und Neues aus der Unterhaltungsmathematik.* It is said by a reviewer in the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* to contain several new topics in this field, and to be written in a very interesting style.

Oldenburg of Munich has published (1918) the eighth edition of *Einführung in die Mathematische Behandlung der Naturwissenschaften,* by W. NERNST and A. SCHOENFLIES, which was intended primarily for chemists. It will be recalled that *The Elements of the Differential and Integral Calculus* by J. W. A. YOUNG and C. E. LINEBARGER (New York, Appleton, 1900) was based upon the second edition of this book (1898) which first appeared in 1895.

The forty-fourth volume (Jahrgang 1913) of *Jahrbuch über die Fortschritte der Mathematik* was completed in 1918 and contained 86 + 1210 pages,—slightly more than the previous volume—The first number of the eightieth volume of the *Mathematische Annalen* was published last August.

A third edition of *American Men of Science, A Biographical Directory,* edited by J. McK. CATTELL, is in course of preparation. The appeal for information